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Projective Differential Geometry of Rational Cubic Curves.

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INTRODUCTION.

In his "Projective Differential Geometry of Curves and Ruled Surfaces" Wilczynski has shown that the general projective theory of plane curves may be based upon the consideration of a linear differential equation of the third order

$$y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0. \tag{1}$$

That this can be done depends essentially upon the fact that the coefficients p_1, p_2, p_3 have the same significance for all integral curves of (1), and that the most general integral curve of (1) is obtained from any particular one by a projective transformation. The invariants of a linear homogeneous differential equation therefore characterize the projective properties of its integral curves.

The object of this paper is to apply to certain particular rational cubic curves the interpretations which Wilczynski has given to the invariants and covariants of equation (1). Before stating the problem more in detail it is necessary to summarize briefly those theories which are to be applied.

The most general transformation which converts (1) into another equation of the same form and order is:

$$x = f(\xi), \quad y = \lambda(\xi)\zeta, \tag{2}$$

where $f(\xi)$ and $\lambda(\xi)$ are arbitrary functions of ξ. All transformations of the form (2) form an infinite continuous group which we shall denote by "G." Any differential equation obtained from (1) by a transformation of the form (2) is said to be equivalent to (1). Any function of p_1, p_2, p_3 and of their derivatives which has the same value as that same function formed from the coefficients of an equivalent equation is called an *absolute invariant.* If such a function contains y or any of its derivatives it is called a *covariant.* An irreducible rational integral expression

$$f\,(y, y', y'', \ldots, p_1, p_2, p_3, p_1', p_2', p_3', \ldots)$$

is called a *relative* covariant if the equation $f=0$ has as its consequence the same equation in the new variables, *i. e.*,

$$f(\zeta, \zeta', \zeta'', \ldots\ldots) = 0.$$

The transformation (2) may be decomposed into two others, a transformation of the dependent variable of the form

$$y = \lambda(x)\zeta \tag{3}$$

followed by a transformation of the independent variable

$$x = f(\xi). \tag{4}$$

All transformations of the form (3) form a subgroup of G. If we make a transformation of this form, equation (1) becomes

$$\zeta''' + 3\pi_1\zeta'' + 3\pi_2\zeta' + \pi_3\zeta = 0, \tag{5}$$

where

$$\left.\begin{aligned} \pi_1 &= \frac{1}{\lambda}(\lambda' + p_1\lambda), \\ \pi_2 &= \frac{1}{\lambda}(\lambda'' + 2p_1\lambda' + p_2\lambda), \\ \pi_3 &= \frac{1}{\lambda}(\lambda''' + 3p_1\lambda'' + 3p_2\lambda' + p_3\lambda). \end{aligned}\right\} \tag{6}$$

Those functions of p_1, p_2, p_3 and of their derivatives which remain invariant under all transformations of this form are called *seminvariants* and *semi-covariants*. The fundamental seminvariants are:

$$P_2 = p_2 - p_1^2 - p_1', \quad P_3 = p_3 - 3p_1p_2 + 2p_1^3 - p_1''. \tag{7}$$

The fundamental semi-covariants are y itself and

$$z = y' + p_1y, \quad \rho = y'' + 2p_1y' + p_2y. \tag{8}$$

Under the transformation (4) equation (1) becomes

$$y''' + 3\bar{p}_1y'' + 3\bar{p}_2y' + \bar{p}_3y = 0, \tag{9}$$

where

$$\bar{p}_1 = \frac{1}{\xi'}(p_1 + \eta), \quad \bar{p}_2 = \left(\frac{1}{\xi'}\right)^2\left(p_2 + \eta p_1 + \frac{1}{3}\mu + \frac{1}{2}\eta^2\right), \quad \bar{p}_3 = \left(\frac{1}{\xi'}\right)^3 p_3, \tag{10}$$

the following abbreviations being also used:

$$\xi' = \frac{d\xi}{dx}, \quad \eta = \frac{\xi''}{\xi'}, \quad \mu = \eta' - \frac{1}{2}\eta^2.$$

The most important relative invariants for our purpose are

$$\theta_3 = P_3 - \frac{3}{2}P_2', \quad \theta_8 = 6\theta_3\theta_3'' - 7(\theta_3')^2 + 27P_2\theta_3^2. \tag{11}$$

It is convenient for some purposes to reduce the equation (1) to certain standard forms: the semi-canonical form, the Laguerre-Forsyth and the Halphen canonical forms. The first is characterized by the absence of the term involving $\frac{d^2y}{dx^2}$. The second is characterized by the absence of the terms involving $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$. The Halphen canonical form is characterized by the conditions $\theta_3 = 1$ and $p_1 = 0$.

As the letter x will be needed to represent a cartesian coordinate, we shall hereafter denote the independent variable by t. Let y_1, y_2, y_3 be the solutions of equation (1). We shall interpret them as the homogeneous coordinates of a point P_y in a plane, the triangle of reference being the x axis, the y axis and the line at infinity, and the unit point being so chosen that $\frac{y_2}{y_1} = x$ and $\frac{y_3}{y_1} = y$ are cartesian coordinates. As t varies, P_y describes a plane curve and the two other points P_z and P_ρ, determined by the semi-covariants (8), describe curves C_z and C_ρ which are semi-covariantly connected with C_y. P_z is a point on the tangent to C_y at P_y, and P_ρ is some point of the plane, not on the tangent if P_y is not a point of inflection. These three points may therefore be taken as the vertices of a triangle of reference. More specifically we define the coordinates of a point with respect to the triangle of reference $P_yP_zP_\rho$ as follows: Any expression of the form

$$\sigma = x_1 y + x_2 z + x_3 \rho$$

assumes three values $\sigma_1, \sigma_2, \sigma_3$ if we substitute for y in succession the three solutions y_1, y_2, y_3. Then $\sigma_1, \sigma_2, \sigma_3$ will be the homogeneous rectangular coordinates of a point P_σ whose homogeneous coordinates with respect to the triangle $P_yP_zP_\rho$ will be proportional to x_1, x_2, x_3, the unit point of the latter system being chosen in an appropriate fashion. The cartesian coordinates of the same point will be $x = \frac{\sigma_2}{\sigma_1}$, $y = \frac{\sigma_3}{\sigma_1}$.

Since a change of the independent variable displaces P_z along the tangent, it follows that unless the independent variable be chosen in some special way, the curve C_z may be thought of as arbitrary. If, however, the differential equation is in the Halphen canonical form, P_z is the intersection of the tangent with the line upon which are situated the three points of inflection of the eight-pointic nodal cubic of P_y. The rectangular homogeneous coordinates of this point, P_c, are given by the covariant expressions*

$$\theta_3' y_i + 3\theta_3 z_i, \qquad (i = 1, 2, 3), \tag{12}$$

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* Wilczynski, "Projective Differential Geometry of Curves and Ruled Surfaces," p. 85.

which, in accordance with the above statement, reduce to $3z_i$ for $\theta_3 = 1$. Its locus will be designated as the curve C_c.

At any point of the curve C_y the equation of the osculating conic, referred to the triangle of reference $P_y P_z P_\rho$, is

$$x_2^2 - 2x_1 x_3 + P_2 x_3^2 = 0. \tag{13}$$

The eight-pointic cubics at any point P_y have a ninth point in common which is called the Halphen point of P_y. It will be denoted by P_h. Its coordinates referred to the triangle $P_y P_z P_\rho$ are given by the expressions*

$$\begin{aligned} x_1 &= 7(5\theta_3'\theta_8 - 756\theta_3^4)^2 + 25\theta_8^3 + 1575\theta_3^2\theta_8^2 P_2, \\ x_2 &= 210\theta_3\theta_8(5\theta_3'\theta_8 - 756\theta_3^4), \\ x_3 &= 3150\theta_3^2\theta_8^2. \end{aligned}$$

The object of this paper may now be stated as follows: Given a rational cubic curve in cartesian coordinates; to set up a linear differential equation of the third order determined by it; to determine the nature of the semi-covariant curves C_z and C_ρ; to find the equation of the osculating conic in cartesian coordinates, study its nature and the locus of its centers; to determine the locus of the Halphen points and the curve C_c.

1. *The Cubical Parabola.*

The cubic equation $y = ax^3$ may be written in the homogeneous form

$$a y_2^3 - y_3 y_1^2 = 0, \tag{1}$$

and, therefore, the curve may be represented parametrically thus:

$$y_1 = 1, \quad y_2 = t, \quad y_3 = at^3, \tag{2}$$

the cartesian coordinates being defined by the equations

$$x = \frac{y_2}{y_1}, \quad y = \frac{y_3}{y_1}.$$

To determine the differential equation

$$y''' + 3p_1 y'' + 3p_2 y' + p_3 y = 0,$$

whose fundamental solutions are y_1, y_2, y_3, we need to solve the following system for p_1, p_2, p_3:

$$y_i''' + 3p_1 y_i'' + 3p_2 y_i' + 3p_3 y_i = 0, \qquad (i = 1, 2, 3). \tag{3}$$

We find

$$p_1 = -\frac{1}{3t}, \quad p_2 = 0, \quad p_3 = 0. \tag{4}$$

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* Wilczynski, *ibid.*, p. 68.

Therefore the theory of the cubic curve $y = ax^3$ may be based upon the equation

$$y''' - \frac{1}{t} y'' = 0. \tag{5}$$

The parametric equations of the curve C_z are

$$z_i = y_i' + p_1 y_i, \qquad (i = 1, 2, 3),$$

or

$$z_1 = -\frac{1}{3t}, \quad z_2 = \frac{2}{3}, \quad z_3 = \frac{8}{3} a t^2. \tag{6}$$

Eliminating t, the equation becomes

$$a z_2^3 - z_3 z_1^2 = 0,$$

which is the original curve itself, although P_y and P_z do not coincide. *The point P_z which corresponds to any point P_y of the cubic is found by drawing the tangent to the cubic at P_y. The second point of intersection of this tangent with the curve is P_z.*

The parametric equations of the curve C_ρ are

$$\rho_i = y_i'' + 2 p_1 y_i' + p_2 y_i, \qquad (i = 1, 2, 3),$$

or

$$\rho_1 = 0, \quad \rho_2 = -\frac{2}{3t}, \quad \rho_3 = 4at. \tag{7}$$

Hence the curve C_ρ is the line at infinity. P_ρ being one of the vertices of the triangle of reference, and being at infinity, it follows that two sides of the triangle of reference $P_y P_z P_\rho$ are always parallel. The slope of the parallel sides is $y : x$ or $\rho_3 : \rho_2 = -6at^2$.

The parametric equations of the curve C_c (which is the locus of P_z when equation (1) is in the Halphen canonical form) are

$$\sigma_i = \theta_3' y_i + 3 \theta_3 z_i, \qquad (i = 1, 2, 3),$$

or

$$\sigma_1 = \frac{60}{27t^3} \cdot \frac{4}{3t}, \quad \sigma_2 = \frac{60}{27t^3} \cdot \frac{1}{3}, \quad \sigma_3 = -\frac{5}{3} \cdot \frac{60}{27t^3} \cdot at^2.$$

Its cartesian equation, obtained by eliminating t, is

$$y = -80ax^3. \tag{8}$$

It is again a cubical parabola.

The equation of the osculating conic at P_y, referred to the particular triangle of reference at that point, is

$$x_2^2 - 2 x_1 x_3 + P_2 x_3^2 = 0.$$

Since

$$P_2 = p_2 - p_1^2 - p_1' = -\frac{4}{9t^2},$$

the equation becomes

$$x_2^2 - 2\,x_1 x_3 - \frac{4}{9\,t^2}\,x_3^2 = 0. \tag{9}$$

In order to find the equation of this conic in cartesian coordinates, we form the three expressions

$$\sigma_i = x_1 y_i + x_2 z_i + x_3 \rho_i, \qquad (i = 1, 2, 3),$$

solve them for x_1, x_2, x_3 and then substitute these values in equation (9).

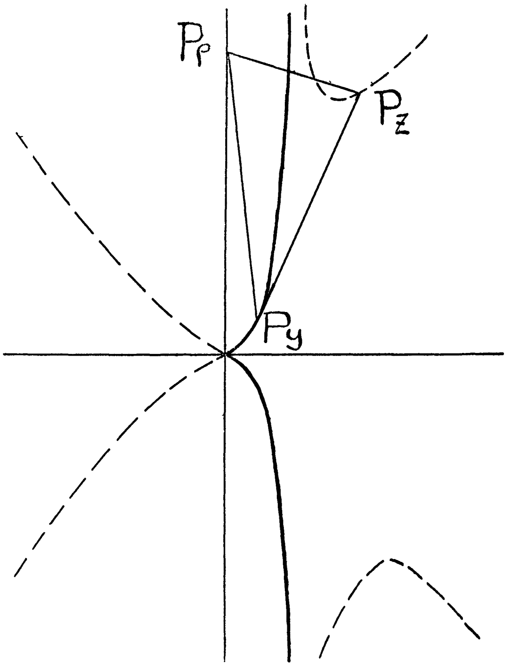

Fig. 1.

Putting $\frac{\sigma_2}{\sigma_1} = x$, $\frac{\sigma_3}{\sigma_1} = y$, we have

$$\frac{x_2}{x_1} = 3 \cdot \frac{6\,at^2x + y - 7\,at^3}{20\,at^2 + 6\,atx + y/t}, \quad \frac{x_3}{x_1} = \frac{9\,t}{2} \cdot \frac{y + 4\,at^3 - 3\,axt^2}{20\,at^2 + 6\,atx + y/t}.$$

The equation of the conic in cartesian coordinates is therefore

$$45\,a^2t^4x^2 - y^2 + 15\,at^2xy - 24\,a^2t^5x - 40\,at^3y + 5\,a^2t^6 = 0. \tag{12}$$

On examining this equation we can see that *the osculating conic never degenerates except at the origin, where it reduces to a pair of coincident lines. Everywhere else it is a hyperbola.*

The centers of the osculating hyperbolas are given by the equations

$$x = \frac{8}{5}\,t, \quad y = -8\,at^3.$$

They are therefore on the curve

$$y = a\left(-\frac{5x}{4}\right)^3, \tag{13}$$

which is again a cubical parabola. (See Fig. 1.) (In the figures, C_y is represented by the unbroken line ———— ; C_z, by — — — ; C_ρ, by ------ ; centers of conics, by — - — - —.)

To find the Halphen point we must first compute the following quantities:

$$7\,(5\,\theta_3'\,\theta_8 - 756\,\theta_3^4)^2 + 25\,\theta_8^3 + 1575\,\theta_3^2\,\theta_8^2\,P^2 = \frac{1}{27^6}\cdot\frac{1}{t^{24}}\cdot(219520)\cdot 10^8,$$

$$210\,\theta_3\,\theta_8\,(5\,\theta_3'\,\theta_8 - 756\,\theta_3^4) = \frac{1}{27^6}\cdot\frac{1}{t^{22}}\cdot(691488)\cdot 10^8,$$

$$3150\,\theta_3^2\,\theta_8^2 = \frac{1}{27^6}\cdot\frac{1}{t^{22}}\,(889056)\cdot 10^8.$$

Its coordinates are therefore

$$x_1 = \frac{\alpha}{t^2}, \quad x_2 = \frac{\beta}{t}, \quad x_3 = \gamma,$$

where

$$\alpha = 219520, \quad \beta = 691488, \quad \gamma = 889056.$$

The rectangular homogeneous coordinates are

$$\sigma_1 = \frac{1}{3t^2}\,(3\alpha - \beta), \quad \sigma_2 = \frac{1}{3t}\,(3\alpha + 2\beta - 2\gamma), \quad \sigma_3 = \frac{t}{3}\,(3\alpha + 8\beta + 12\gamma),$$

so that

$$x = -8\,t, \quad y = -512\,a t^3. \tag{14}$$

These two equations show that P_h is again a point of the cubic, as the general notion of an Halphen point enables us to predict. From the signs of the coordinates it follows that P_h is always in the opposite quadrant from P_y. The class of points consisting of successive Halphen points is not a closed class, since the x coordinate of each P_h is 64 times as great as the second preceding one, and so the original point can never be reached.

2. *The Semi-cubical Parabola.*

The curve $y^2 = \frac{1}{a}x^3$ or $a\,y_3^2\,y_1 - y_2^3 = 0$ can be represented parametrically by the equations

$$y_1 = 1, \quad y_2 = a t^2, \quad y_3 = a t^3. \tag{1}$$

The differential equation whose solutions are y_1, y_2, y_3, is

$$y''' - \frac{2}{t}\,y'' + \frac{2}{t^2}\,y' = 0. \tag{2}$$

Using the methods of the previous paragraph, we find the equations of the curves C_z and C_ρ to be respectively

$$y^2 = -\frac{49}{32a} x^3, \tag{3}$$

$$x = 0. \tag{4}$$

The equation of C_c becomes

$$y^2 = -\frac{80}{a} x^3, \tag{5}$$

which is again a semi-cubical parabola.

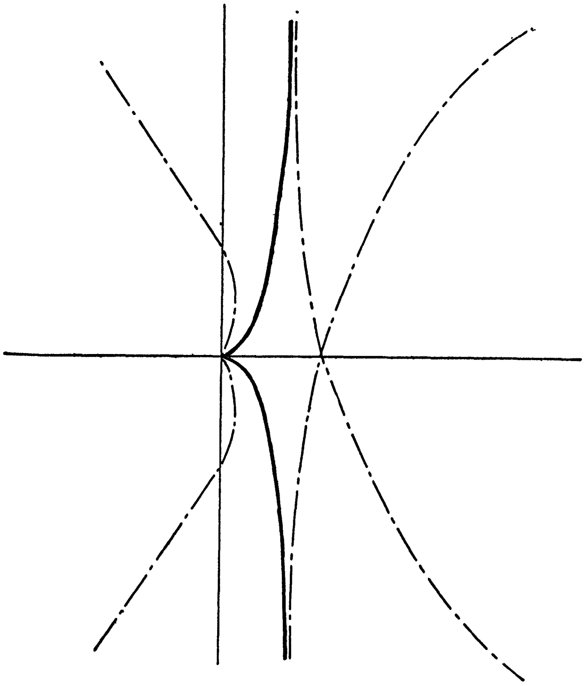

FIG. 2.

The equation of the osculating conic is

$$45\, t^2 x^2 + 5\, y^2 - 24\, txy + 15\, t^4 ax - 40\, at^3 y - a^2 t^6 = 0, \tag{6}$$

which is an ellipse except at the origin, where it degenerates into a pair of coincident straight lines.

The centers of these conics lie on a semi-cubical parabola

$$y^2 = \frac{32}{5a} x^3. \quad \text{(See Fig. 2.)} \tag{7}$$

The cartesian coordinates of the Halphen point are

$$x = a\left(\frac{t}{8}\right)^2, \quad y = a\left(\frac{-t}{8}\right)^3.$$

From the choice of parameters in equation (1), at^2 is the x coordinate of P_y. The x of P_h is then 1/64 of the x of P_y. Hence the class of successive Halphen points of P_y is not closed, as the original point is never reached again. As in the case of the cubic $y = ax^3$, P_h always lies in a different quadrant from P_y.

3. *The Cissoid of Diocles.*

The equations of the cissoid of Diocles in cartesian and homogeneous coordinates are, respectively,

$$y^2 = \frac{1}{b} \cdot \frac{x^3}{2a - x}, \quad (a > 0,\ b > 0),$$

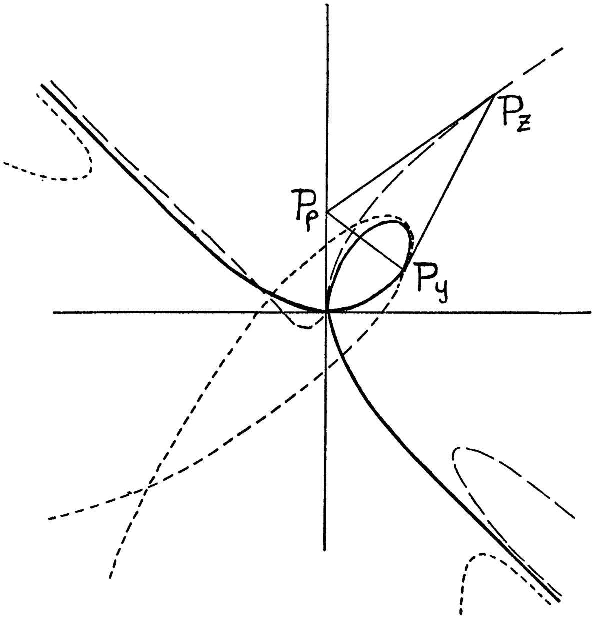

FIG. 3.

and

$$2ab\, y_3^2 y_1 - b\, y_3^2 y_2 - y_2^3 = 0;$$

so that we may put

$$y_1 = 1 + bt^2, \quad y_2 = 2abt^2, \quad y_3 = 2abt^3. \tag{1}$$

The differential equation whose fundamental solutions are y_1, y_2, y_3 is

$$y''' - \frac{2}{t} y'' + \frac{2}{t^2} y' = 0. \tag{2}$$

The equations of the curves C_z and C_ρ are

$$y^2 = -\frac{49}{32b} \cdot \frac{x^3}{2a - x}, \tag{3}$$

$$x = 0. \tag{4}$$

The equation of C_c *becomes*

$$y^2 = -\frac{80}{b} \cdot \frac{x^3}{2a + x}, \tag{5}$$

and this is again a cissoid. (See Fig. 3.)

The equation of the osculating conic is

$$x^2t^2(b^2t^4+15\,bt^2-45)-5\,y^2-8\,txy\,(5\,bt^2-3)-2\,xct^4(2\,bt^2+15)$$
$$+80\,cyt^3+4\,c^2t^6=0,\ \ (c=ab). \qquad (6)$$

At the origin the osculating conic degenerates, and for $x=a\frac{1+\sqrt{5}}{3}$ *it is a parabola. For* $0<x<a\frac{1+\sqrt{5}}{3}$ *the conics are ellipses, and for* $x>a\frac{1+\sqrt{5}}{3}$ *they are hyperbolas.*

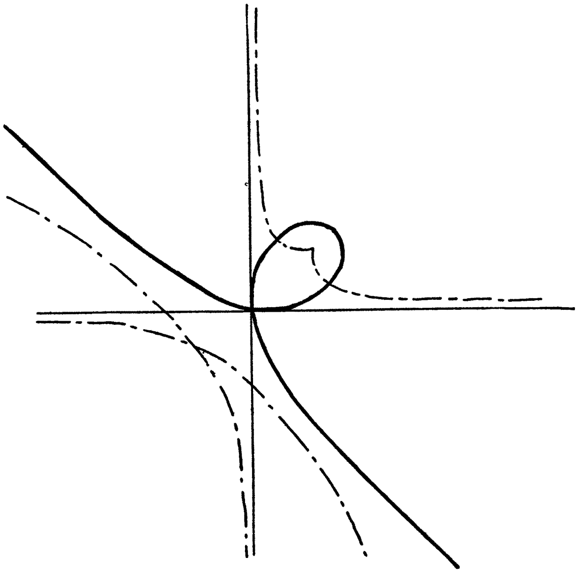

FIG. 4.

The center of the osculating conic has the coordinates

$$x=5\,ct^2\frac{2\,bt^2-1}{5\,b^2t^4-5\,bt^2-1},\quad y=4\,ct^3\cdot\frac{5+bt^2}{5\,b^2t^4-5\,bt^2-1}.$$

Its locus is therefore a unicursal quintic. (See Fig. 4.)

The cartesian coordinates of P_h are

$$x=abt^2\frac{6\,\alpha+8\,\beta}{3\,\alpha\,(1+bt^2)-2\,\beta\,(1-2\,bt^2)+2\,\gamma},$$
$$y=abt^3\frac{6\,\alpha+14\,\beta+16\,\gamma}{3\,\alpha\,(1+bt^2)-2\,\beta\,(1-2\,bt^2)+2\,\gamma},$$

where

$$\alpha=3380608,\quad \beta=-2469600,\quad \gamma=889056.$$

Between these there is the relation

$$\frac{y}{x}=t\cdot\frac{6\,\alpha+14\,\beta+16\,\gamma}{6\,\alpha+8\,\beta}=-\frac{t}{8}.$$

Consequently, to find the Halphen point of P_y, draw the line through the origin with the slope $-t/8$. Its second intersection with the given curve C_y is the desired point P_h. We may note that here, as in the previous two cases, P_h never lies in the same quadrant as P_y, and that the class of successive Halphen points of P_y is not closed. The latter remark also follows from the fact that these three curves are projectively equivalent to each other.

4. *The Folium of Descartes.*

We choose the parametric representation

$$y_1 = 1 + t^3, \quad y_2 = at, \quad y_3 = at^2$$

to represent the folium whose cartesian equation is

$$x^3 + y^3 - axy = 0.$$

The parametric equations of C_z are

$$x = \frac{a}{2t^2} \cdot \frac{1}{1+t^3}, \quad y = \frac{a}{2t} \cdot \frac{2+t^3}{1+t^3}.$$

The locus of C_ρ is, as in the previous case,

$$x = 0.$$

The parametric equations of C_c are

$$x = \frac{a}{1+t^3} \cdot \frac{-2t^6 + 5t^3 - 1}{2t^2}, \quad y = \frac{a}{1+t^3} \cdot \frac{-t^6 + 5t^3 - 2}{2t},$$

a unicursal sextic.

The equation of the osculating conic at P_y is

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where

$$A = 1 + 10t^3 + 5t^6 + 5t^9, \quad B = t(5 + 5t^3 + 10t^6 + t^9), \quad H = -t^2(5 + t^3 + t^6),$$
$$G = -\frac{at^4}{2}(5 + t^6), \qquad F = -\frac{a}{2}(1 + 5t^6), \qquad C = a^2 t^5.$$

The conic is degenerate only for $t = -1$, *i. e.*, at infinity. *For all points of the loop the osculating conic is an ellipse; at the origin it is a parabola; and for all other points it is a hyperbola.*

The locus of the centers of the conics is given by the equations

$$x = \frac{at}{10} \cdot \frac{5 + t^3}{1 + t^3}, \quad y = \frac{a}{10t} \cdot \frac{1 + 5t^3}{1 + t^3}.$$

It is a unicursal quartic.

The cartesian coordinates of P_h are found to be

$$x = \frac{at}{1+t^3} \cdot \frac{t^{15}}{(1-t^3)(t^{18}+t^{12}+t^6+1)}, \quad y = \frac{at^2}{1+t^3} \cdot \frac{-t^6}{(1-t^3)(t^{18}+t^{12}+t^6+1)},$$

which satisfy the equation $x^3 + y^3 - axy = 0$, as they should. From the above equation follows the relation

$$x = -t^8 y.$$

Consequently, for any value of t, the signs of x and y are opposite. *Therefore P_h can never lie in the first quadrant.*

CHICAGO, ILL.